

# Critical velocity of a clean one-dimensional superconductor

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(Received 15 April 2009; revised manuscript received 15 September 2009; published 8 October 2009)

We revisit the problem of critical velocity of a clean one-dimensional superconductor. At the level of mean-field theory, we find that the zero-temperature value of the critical velocity—the uniform velocity of the superfluid condensate at which the superconducting state becomes unstable—is a factor of  $\sqrt{2}$  smaller than the Landau critical velocity. This is in contrast to a prior finding, which held that the critical velocity is equal to the Landau critical velocity. The smaller value of the critical velocity, which our analysis yields, is the result of a pre-emptive Clogston-Chandrasekhar-type discontinuous phase transition, and is an analog of the threshold value of the uniform exchange field of a superconductor previously investigated by Sarma [J. Phys. Chem. Solids **24**, 1029 (1963)] and by Maki and Tsuneto [Prog. Theor. Phys. **31**, 945 (1964)]. We also consider the impact of nonzero temperature, study critical currents, and examine metastability and its limits in the temperature versus flow-velocity phase diagram. In addition, we comment on the effects of electron scattering by impurities.

DOI: [10.1103/PhysRevB.80.134507](https://doi.org/10.1103/PhysRevB.80.134507)

PACS number(s): 74.25.Sv, 74.62.-c, 74.25.Dw

## I. INTRODUCTION

The Landau criterion<sup>1</sup> concerns the threshold velocity  $v_L$  of an obstacle moving through stationary superfluids at zero temperature beyond which excitations are created and superfluidity is lost. By Galilean invariance, the criterion also applies to the threshold velocity  $v_L$  of a uniformly flowing superfluid; see, e.g., Ref. 2. The value of the threshold velocity  $v_L$  is determined by the following formula

$$v_L = \min_p \left( \frac{E_p}{p} \right), \quad (1)$$

where  $E_p$  is the excitation spectrum. For superconductors the Landau criterion gives  $v_L = \Delta_0/p_F$ , where  $2\Delta_0$  is the pairing gap at zero temperature and zero flow, and  $p_F = \hbar k_F$  is the Fermi momentum of the entities that are paired (with  $k_F$  being the associated angular wave number). However, it was found by Rogers<sup>3-5</sup> that in three spatial dimensions the critical velocity  $v_c$ —the uniform velocity of the superfluid condensate at which the superconducting state becomes unstable—of a clean superconductor exceeds  $v_L$ . Furthermore, as discussed, e.g., in Ref. 5, for superflow velocities  $v$  in the range of  $v_L \leq v \leq v_c$ , gapless excitations occur in clean superconductors. Moreover, the ratio  $v_c^{(3D)}/v_L$  has been found to be  $e/2$  ( $\approx 1.359$ ). In contrast, it was found<sup>5</sup> that in two dimensions  $v_c^{(2D)}/v_L = 1$ . As for the case of one dimension, Bagwell reported<sup>6</sup> that  $v_c^{(1D)}/v_L = 1$ , as found in two dimensions. In contrast to the case of three dimensions, in neither one nor two dimensions is gapless superconductivity predicted to occur in the presence of flow.

In this paper, we reanalyze the critical velocity of a clean superconductor in one spatial dimension via a mean-field treatment, and obtain the dependence of this velocity on the temperature  $T$ , along with the associated temperature-

velocity phase diagram. In particular, we find that at zero temperature the critical velocity is smaller by a factor of  $\sqrt{2}$  than the Landau critical velocity  $v_L$ . Even though for  $v = v_L/\sqrt{2}$  a gap in the quasiparticle excitation spectrum remains, the superconducting state first becomes unstable there. This is in contrast to a previous report,<sup>6</sup> which held that the critical velocity is  $v_L$ , i.e., the velocity at which the gap in the quasiparticle spectrum closes and excitations proliferate so as to destroy superconductivity. The smaller value of the critical velocity, which we obtain here, is the result of a pre-emptive Clogston-Chandrasekhar-type discontinuous phase transition,<sup>7</sup> and is analogous to the threshold uniform exchange field in a superconductor previously investigated by Sarma<sup>8</sup> and by Maki and Tsuneto (SMT).<sup>9</sup> At low temperatures (i.e., for  $T < T^* \approx 0.556T_c^0$ , where  $T_c^0$  is the critical temperature in the absence of flow), the transition from the superconducting to the normal state, occurring due to the presence of flow, remains discontinuous. In contrast, for  $T \geq T^*$  the transition is continuous, i.e., it is associated with a continuously vanishing order parameter, e.g., as  $T$  approaches the flow-velocity-dependent transition temperature  $T_c(v)$  from below.

From the experimental standpoint it is usually more convenient to control the condensate *current* rather than its *velocity*. (An exception is provided by a closed loop threaded by magnetic flux.) The more widely appropriate physical observable is thus the critical current, i.e., the maximum equilibrium current that a superconductor can sustain, and this is what experiments on superconductors frequently measure. It should, however, be remarked that recent experiments on trapped condensates of atomic gases make it possible to probe the critical velocity directly.<sup>10</sup>

Langer and Fisher<sup>11</sup> introduced a fresh perspective on the issue of the critical velocity of a superfluid. They explained

that—as a matter of principle—superflow is inherently unstable even at velocities below those that allow quasiparticle excitations to proliferate. This instability results from the possible occurrence of intrinsic, topologically allowed, collective excitation events, in which, e.g., a vortex ring nucleates locally, traverses an Arrhenius energy barrier, and grows without bound, thus eradicating a quantum of flow velocity. In the case of narrow channels such as the one-dimensional (1D) superconductors discussed here, the fluctuations take the form of phase slip events, which can be either thermal<sup>12</sup> or quantum.<sup>13</sup> But before one takes into account the effects of fluctuations it is important to understand and clarify the behavior at the mean-field level, which is mainly what we consider in the present paper. In other words, the work reported here assumes that the rate of such topological fluctuations is negligibly small, so that our critical-velocity results established on the basis of thermodynamic stability and the possibility of quasiparticle production retain a use. In the present paper we focus on issues of phases of thermodynamic equilibrium, competitions between them, and metastability; we do not attempt to address issues of kinetics such as the rates at which phase transitions proceed and metastable states evolve into stable ones.

The remainder of the Paper is organized as follows. In Sec. II we diagonalize the Hamiltonian of a superconductor in the presence of flow and in Sec. III we calculate the free energy of this system. In Sec. IV we use the self-consistency of the order parameter to calculate the dependence of the order parameter on the flow velocity at various temperatures. In Sec. V we calculate the dependence of the transition temperature on the flow velocity, assuming the transition to be continuous, i.e., in the limit of linear instability. Double solutions for the transition temperature turn out to exist for larger values of the flow velocity. We resolve this issue in Sec. VI by identifying the globally stable solution, i.e., the one that corresponds to the lower free energy. In Sec. VII we address the existence of metastable solutions and discuss the extent to which these provide a superflow-based analog of the phenomena of supercooling and superheating. We also obtain the analog of the superheating limit together with the equilibrium phase boundary and the supercooling limit. Having ascertained the true, equilibrium, velocity-dependent order parameter, we calculate in Sec. VIII the dependence of the supercurrent on the flow velocity, and hence determine the critical current and superfluid density. In Sec. IX we briefly discuss the effect of disorder on the nature of the transition. In Sec. X we contrast the critical-velocity results for one dimension with those for two and three dimensions. We conclude in Sec. XI.

## II. HAMILTONIAN

We shall use a microscopic approach to discuss an effectively one-dimensional superconducting system, in which there is spatially uniform flow at velocity  $v \equiv q/m$ . Here,  $m$  is the electron mass and  $q$  is the corresponding momentum associated with the flow (or equivalently, the angular wave number as we shall set  $\hbar = 1$ ). (What we mean by “effectively one-dimensional” is that the transverse dimensions are much

smaller than the zero-temperature coherence length  $\xi_0$ .) We begin by writing down a Hamiltonian that is equivalent to Eq. (III.4) of Rogers<sup>3</sup> and Eq. (1) of Nozières and Schmitt-Rink<sup>14</sup>

$$H = \sum_{k,\sigma} \left( \frac{(k+q)^2}{2m} - \mu \right) c_{k+q,\sigma}^\dagger c_{k+q,\sigma} + \sum_{k_1,k_2} V_{k_1,k_2} c_{k_1+q,\uparrow}^\dagger c_{-k_1+q,\downarrow} c_{-k_2+q,\downarrow} c_{k_2+q,\uparrow}, \quad (2)$$

where  $c_{k\sigma}$  and  $c_{k\sigma}^\dagger$  are, respectively, the annihilation and creation operators of electrons of spin projection  $\sigma = \uparrow$  or  $\downarrow$  at (one-dimensional) momentum  $k$ ,  $V_{kk'}$  is the BCS pairing potential,<sup>15</sup> and  $\mu$  is the chemical potential. Following Rogers,<sup>3</sup> we assume that  $V_{kk'}$  depends on the difference between  $k$  and  $k'$ , and hence is independent of  $q$ . To justify our ignoring any  $q$  dependence, we note that the critical momentum  $q$  is no bigger than roughly  $mv_L$ , and this is roughly  $(a/\xi_0)k_F$ , i.e., several orders of magnitude smaller than  $k_F$ , where  $a$  is the atomic spacing and  $\xi_0$  is the zero-temperature superconducting coherence length. This form of Hamiltonian anticipates that any Cooper pairs are formed via the pairing of electrons in states  $(k+q, \uparrow)$  and  $(-k+q, \downarrow)$ , and that the resulting pairs have center-of-mass momentum  $2q$ . Next, we make the mean-field approximation and adopt the BCS form for  $V_{kk'}$ ,<sup>15</sup> thus arriving at the Hamiltonian

$$H = \frac{|\Delta|^2}{g} + \sum_k (\varepsilon_q(k) c_{k+q,\uparrow}^\dagger c_{k+q,\uparrow} + \varepsilon_q(-k) c_{-k+q,\downarrow}^\dagger c_{-k+q,\downarrow} - (\Delta^* c_{-k+q,\downarrow} c_{k+q,\uparrow} + \Delta c_{k+q,\uparrow}^\dagger c_{-k+q,\downarrow}^\dagger)), \quad (3)$$

in which  $g = |V_{kk'}|$  is the magnitude of  $V_{kk'}$  in the momentum range for which the pairing potential is nonzero,  $\varepsilon_q(\pm k) \equiv (q \pm k)^2/2m - \mu$ , and  $\Delta = -\sum_{k'} V_{k,k'} \langle c_{-k'+q,\downarrow} c_{k'+q,\uparrow} \rangle$  is the self-consistency condition on the order parameter (and similarly for  $\Delta^*$ ). This mean-field Hamiltonian can be solved via the Bogoliubov-Valatin transformation

$$c_{k+q,\uparrow} = u_k \gamma_{1;k} + v_k \gamma_{2;k}^\dagger \quad \text{and} \quad c_{-k+q,\downarrow} = u_k \gamma_{2;k} - v_k \gamma_{1;k}^\dagger, \quad (4)$$

with

$$u_k^2 = \frac{1}{2} \left( 1 + \frac{\bar{\varepsilon}_k}{E_k} \right) \quad \text{and} \quad v_k^2 = \frac{1}{2} \left( 1 - \frac{\bar{\varepsilon}_k}{E_k} \right), \quad (5)$$

where, for convenience, we have made the definitions  $\bar{\varepsilon}_k \equiv [\varepsilon_q(k) + \varepsilon_q(-k)]/2 = k^2/2m - (\mu - q^2/2m)$  and  $E_k \equiv \sqrt{\bar{\varepsilon}_k^2 + |\Delta|^2}$ . We choose  $\Delta$  to be real and non-negative, and thus may drop the absolute value on  $\Delta$  in the definition of  $E_k$ . With this procedure, the Hamiltonian becomes diagonal

$$H = \frac{\Delta^2}{g} + \sum_k [(E_k + kv) \gamma_{1;k}^\dagger \gamma_{1;k} + (E_k - kv) \gamma_{2;k}^\dagger \gamma_{2;k}] + \sum_k (\bar{\varepsilon}_k - E_k). \quad (6)$$

This expression also holds for higher dimensions, provided we replace  $k$  by  $\vec{k}$ ,  $v$  by  $\vec{v}$ , and  $kv$  by  $\vec{k} \cdot \vec{v}$ .

### III. FREE ENERGY

As the Hamiltonian has been diagonalized, we can readily evaluate the Helmholtz free energy  $F$  of the system as

$$F \equiv \langle H \rangle - TS$$

$$= \langle H \rangle + T \sum_k [f_{1;k} \ln f_{1;k} + (1 - f_{1;k}) \ln(1 - f_{1;k}) + f_{2;k} \ln f_{2;k} + (1 - f_{2;k}) \ln(1 - f_{2;k})], \quad (7)$$

where  $f_1$  and  $f_2$  are Fermi distribution functions defined via

$$f_{1;k} \equiv \langle \gamma_{1;k}^\dagger \gamma_{1;k} \rangle = \frac{1}{e^{\beta(E_k + kv)} + 1} \quad \text{and} \quad f_{2;k} \equiv \langle \gamma_{2;k}^\dagger \gamma_{2;k} \rangle = \frac{1}{e^{\beta(E_k - kv)} + 1}. \quad (8)$$

$\langle \dots \rangle$  indicates an average, weighted by the equilibrium density matrix, and  $\beta \equiv 1/T$  is the inverse temperature (with Boltzmann's constant set to unity). We therefore arrive at the result

$$F = \frac{\Delta^2}{g} + \sum_k \bar{\epsilon}_k - T \sum_k \ln(2 \cosh \beta E_k + 2 \cosh \beta kv). \quad (9)$$

One may equally well calculate the partition function  $Z$  so as to obtain the free energy via  $F = -k_B T \ln(Z)$ . The order parameter is to be determined self-consistently via

$$\Delta = g \sum_k \langle c_{-k+q,\downarrow} c_{k+q,\uparrow} \rangle = g \sum_k \frac{\Delta}{2E_k} (1 - f_{1;k} - f_{2;k}). \quad (10)$$

We note that because the momentum sum runs over both positive and negative values one can safely replace  $f_{1;k}$  by  $f_{2;k}$  (or vice versa) in this equation and thus arrives at the result

$$\Delta = g \sum_k \frac{\Delta}{2E_k} (1 - 2f_{2;k}), \quad (11)$$

which is identical to Eq. (21) of Ref. 6. This equation can also be derived by demanding the stationary nature of the free energy, i.e.,  $\delta F / \delta \Delta = 0$ . The above results, Eqs. (7)–(11) hold for two- and three-dimensional systems as well, provided we replace  $k$  by  $\vec{k}$ ,  $v$  by  $\vec{v}$ , and  $kv$  by  $\vec{k} \cdot \vec{v}$ , but in the following we shall focus mainly on the case of one dimension.

### IV. ORDER PARAMETER

From Eq. (10) or equivalently Eq. (11) we can determine the values of the order parameter at any temperature and superflow velocity. Of course, there is always the trivial solution  $\Delta = 0$ , which corresponds to the normal state. However, in this section our focus is on nontrivial solutions.

#### A. Zero temperature

Let us first examine the limit of zero temperature. In this limit, the self-consistency condition [Eq. (11)] becomes

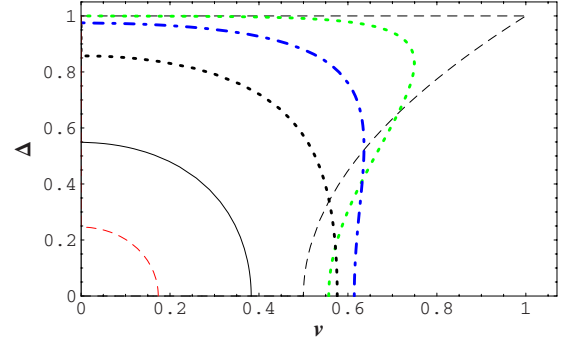


FIG. 1. (Color online) Self-consistent solutions for the order parameter  $\Delta$  (normalized to the zero-temperature, zero-flow value  $\Delta_0$ ) vs superfluid velocity  $v$  (in units of  $v_L$ ) at various temperatures  $T = (0, 0.223, 0.445, 0.668, 0.890, 0.980)T_c^0$  (from top right to bottom left). Note the multivaluedness of  $\Delta$ , which occurs for the lowest three values of the temperatures. It turns out that the lower branches of the order parameter have the maximum in the free energy compared to the upper branches and the trivial  $\Delta = 0$  solution.

$$\Delta(T, v)|_{T=0} = g \sum_{k>0} \frac{\Delta}{E_k} [1 - \Theta(kv - E_k)], \quad (12)$$

where  $\Theta(x)$  is the Heaviside step function. By linearizing the  $k$ -dependent spectrum around  $k = k_F$  and exchanging  $g$  for the  $(T, v) = (0, 0)$  value of the order parameter  $\Delta_0$  we arrive at the following result for  $\Delta(T, v)|_{T=0}$

$$\ln\left(\frac{\Delta}{\Delta_0}\right) = - \int_0^{\omega_D} d\xi \frac{1}{\sqrt{\xi^2 + \Delta^2}} \Theta(k_F v - \sqrt{\xi^2 + \Delta^2}). \quad (13)$$

This condition yields two branches of solutions for the order parameter: (1) If  $k_F v < \Delta$  we obtain  $\Delta = \Delta_0$ ; on the other hand (2) if  $k_F v > \Delta$  the condition becomes

$$\ln\left(\frac{\Delta}{\Delta_0}\right) = - \int_0^{\sqrt{k_F^2 v^2 - \Delta^2}} \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} = - \sinh^{-1}\left(\frac{\sqrt{k_F^2 v^2 - \Delta^2}}{\Delta}\right), \quad (14)$$

which leads to  $\Delta^2 = 2k_F v \Delta_0 - \Delta_0^2$ . These two zero-temperature solutions were first obtained by Sarma<sup>8</sup> in connection with the exchange-field effect in superconductors. In Fig. 1 we show these two zero-temperature solutions [i.e., the horizontal and parabolic curves emanating from the upper right point (1,1)] together with the corresponding solutions at nonzero temperatures. By continuity, it is not surprising that there is a low-temperature regime in which there is multivaluedness in the solutions for  $\Delta$ , as we shall see in Sec. IV B and in Fig. 1.

#### B. Nonzero temperatures

For these, we can solve Eq. (10) or equivalently Eq. (11) numerically and obtain the velocity-dependent order parameter  $\Delta(T, v)$  at arbitrary temperatures, as illustrated in Fig. 1 for several values of the temperature. We remark that in solving for the order parameter we have linearized the spectrum  $E_k \pm kv$  about Fermi momentum  $k_F$ ; this linearization is valid as long as  $\Delta_0 / E_F \ll 1$ . From Fig. 1 it is evident that over a

certain (higher) range of velocities  $v$  there are two solutions for  $\Delta$ , these solutions differ from those obtained by Bagwell,<sup>6</sup> by whom multiple solutions were not found. This type of multiplicity feature was first observed by Sarma<sup>8</sup> in the context of the exchange field in superconductors. In fact, the self-consistency conditions holding in the exchange-field case are identical to the ones holding here, provided one makes an identification between  $k_F v$  and the exchange-field energy  $\mu_B h$ . The two situations appear to be essentially equivalent.

The feature of double solutions that we have found for the order parameter, as shown in Fig. 1, is in contrast with previous findings<sup>6</sup> (see, in particular, Fig. 3 therein). We suspect that this discrepancy results from the use of an iterative scheme for solving the self-consistency equation numerically; when multiple solutions exist, such scheme yields results that depend sensitively on the initial conditions. We have instead used the bisection method, which locates all possible solutions.

## V. TRANSITION TEMPERATURES: LINEAR INSTABILITY

If one assumes that the transition between the superconducting and normal states is continuous, one can determine the transition temperature via the self-consistency condition taken in the limit of vanishing  $\Delta$ . We note that, owing to the relation of the Fermi function to the Matsubara sum (see, e.g., Ref. 5), Eq. (10) is equivalent to

$$\Delta = g \sum_k T \sum_{\omega_n} \frac{-\Delta}{(i\omega_n - kv - E_k)(i\omega_n - kv + E_k)}, \quad (15)$$

where  $\omega_n \equiv 2\pi T[n + (1/2)]$  with  $n=0, \pm 1, \pm 2, \dots$ , are the Matsubara frequencies. We denote by  $T_c$  the value of the temperature that solves Eq. (15) in the limit  $\Delta \rightarrow 0$ . In this limit, imposing a cutoff  $\omega_D$  on the Matsubara frequencies and integrating out the momentum  $k$ , we arrive at the condition

$$1 = g N_0 2\pi T_c \sum_{\omega_n > 0}^{\omega_D} \text{Re} \left( \frac{1}{\omega_n + ik_F v} \right), \quad (16)$$

in which we have redefined  $\omega_n$  to mean  $\omega_n = 2\pi T_c [n + (1/2)]$ . By exchanging  $g$  for the zero-flow critical temperature  $T_c^0$  and performing the summation Eq. (16) becomes

$$\ln \left( \frac{T_c}{T_c^0} \right) = \psi \left( \frac{1}{2} \right) - \text{Re} \psi \left( \frac{1}{2} + \frac{ik_F v}{2\pi T_c} \right), \quad (17)$$

where  $\psi(x)$  is the di-gamma function.<sup>16</sup>

The solutions of Eq. (17) for  $T_c$  are shown in Fig. 2, which, like Fig. 1, exhibits a double-solution feature for a range of velocities. The solution at  $T_c=0$  and  $k_F v = \Delta_0/2$  corresponds to the branch  $\Delta^2 = 2k_F v \Delta_0 - \Delta_0^2$ . As we shall see below, this branch corresponds to an *unstable* solution; a correct description of the  $T_c$  vs  $v$  phase diagram requires the consideration of free energies.

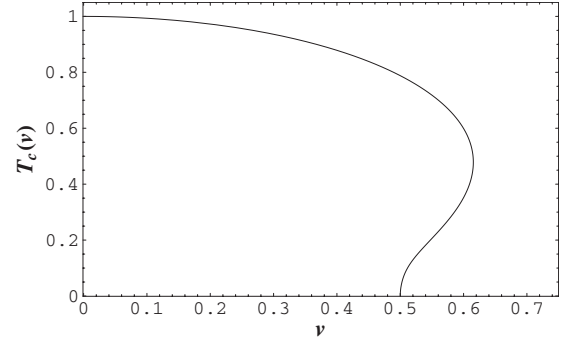


FIG. 2. Critical temperature (from linear instability)  $T_c(v)$  (in unit of  $T_c^0$ ) vs superfluid velocity  $v$  (in units of  $v_L$ ) as obtained from Eq. (17). Note the occurrence of the unphysical multivaluedness.

## VI. VELOCITY-TEMPERATURE PHASE DIAGRAM

In the light of the results obtained in Sec. V we ask the question: what is the true transition temperature? To determine this, one has to take into account the multiple solutions of the order-parameter self-consistency condition and compare their free energies (e.g., see Ref. 4). With this prescription we ask: will our transition temperature  $T_c(v)$  have the same functional form as the  $T_c(h)$  of Sarma<sup>8</sup> and of Maki and Tsuneto,<sup>9</sup> in which  $h$  is the magnetic exchange field?

In the case considered by SMT, a comparison is made between the free energies of a spin-unpolarized superconducting state and a normal state that is partially spin-polarized, both states being subject to an exchange field. In the present case the comparison is between a flowing superconducting state and a stationary normal state. Viewed from a reference frame that flows with the superconducting state, the normal state flows and can be regarded as the analog of SMT's polarized state. Furthermore, our superconducting state is the analog of SMT's superconducting state. This perspective suggests that the correct procedure for our purposes is to compare the free energies of the flowing superconducting state and the stationary normal state, as was first done by Bagwell.<sup>6</sup>

The physical reason for the equivalence between the exchange-field case considered by Sarma and the superflow case considered here is as follows. In the case of Sarma, due to the exchange field, the up and down electrons in a Cooper pair have an energy difference of  $\mu_B B$ . In the case of flow, the electron pairing is between states  $(k+q)\uparrow$  and  $(-k+q)\downarrow$ , and which have an energy difference of  $2k_F q/m$  near the Fermi surface. In the former case, the normal state can be polarized, and thus can reduce its energy by an amount proportional to  $h^2$ . In the latter case the stationary normal state has an energy that is lower than the flowing normal state by an amount proportional to  $q^2$ . Hence, the two scenarios appear to be equivalent.

To understand the root of our state-selection procedure, we begin with an analogy. Consider the liquid and crystalline states of a particular material. Following Callen,<sup>17</sup> we observe that the information sufficient to specify uniquely an equilibrium state is greater for the crystalline state than for the liquid state, owing to the spontaneously broken translational symmetry of the crystal and the resulting low-energy

Goldstone field, i.e., the displacement field. Thus, when assessing the relative thermodynamic stability of a liquid and a crystalline state, one must specify a displacement field for the crystal but one must not (and indeed cannot) do so for the liquid state. It is meaningful to speak of an equilibrium crystalline state that supports a specified static shear stress, but not so for a liquid, as stress would induce a nonequilibrium dissipative steady state of shear flow, in which entropy would constantly be being produced. In summary, there is a unique equilibrium state for the liquid, but there is a family of equilibrium states for the crystal, differing in their displacement fields (or strain field, or its thermodynamic conjugate, the stress field). One is entitled to consider the relative thermodynamic stability of a liquid and any one of this family of crystalline states. Returning now to the case of superconductivity, the precise analogy is on the one hand between the crystal and the superconductor and on the other hand between the liquid and the normal metal.

One is thus entitled to consider the relative thermodynamic stability of a stationary normal state and any one of the family of the flowing superconducting states. That there is a family of flowing superconducting states originates in the spontaneously broken gauge symmetry of the superconducting state, which amounts to a controllable thermodynamic field, (i.e., the phase field), which is the analog of the displacement field of the crystal. (The analogy is: phase  $\leftrightarrow$  displacement; velocity  $\leftrightarrow$  strain; current density  $\leftrightarrow$  stress.) By contrast, the normal state is unique.

To obtain the phase diagram, one should then compare the free energy  $F_s(q, \mu)$  of a flowing superconducting state to the free energy  $F_n(0, \mu)$  of a stationary (i.e., not flowing) normal state.<sup>6</sup> The energy difference between the flowing and stationary normal states plays the role of paramagnetic energy. We shall see that the true equilibrium transition temperature then has exactly the functional form obtained by Sarma<sup>8</sup> and by Maki and Tsuneto.<sup>9</sup>

The free energy [Eq. (9)] of the superconducting state at a fixed chemical potential  $\mu$  is given explicitly by

$$F_s(q, \mu) = \frac{\Delta^2}{g} + \sum_k \left( \frac{k^2}{2m} + \frac{q^2}{2m} - \mu \right) - T \sum_k \ln \left[ 2 \cosh \beta \sqrt{\left( \frac{k^2}{2m} + \frac{q^2}{2m} - \mu \right)^2 + \Delta^2} + 2 \cosh \beta kv \right]. \quad (18)$$

The stationary, normal state at chemical potential  $\mu - (q^2/2m)$  thus has free energy

$$F_n\left(0, \mu - \frac{q^2}{2m}\right) = \sum_k \left( \frac{k^2}{2m} + \frac{q^2}{2m} - \mu \right) - T \sum_k \ln \left[ 2 \cosh \beta \left( \frac{k^2}{2m} + \frac{q^2}{2m} - \mu \right) + 2 \right]. \quad (19)$$

Our goal is to find the difference between two energies at the same chemical potential, which is given by

$$F_s(q, \mu) - F_n(0, \mu) = \left[ F_s(q, \mu) - F_n\left(0, \mu - \frac{q^2}{2m}\right) \right] - \left[ F_n(0, \mu) - F_n\left(0, \mu - \frac{q^2}{2m}\right) \right], \quad (20)$$

where the term in the second pair of square brackets equals  $-Nq^2/2m$  (i.e., the analog of the paramagnetic energy), with  $N$  being the total number of electrons. Hence, from Eqs. (19) and (20) we arrive at the result that

$$F_s(q, \mu) - F_n(0, \mu) = N \frac{q^2}{2m} + \frac{\Delta^2}{g} - T \sum_k \ln \left[ \frac{\cosh \beta \sqrt{\left( \frac{k^2}{2m} + \frac{q^2}{2m} - \mu \right)^2 + \Delta^2} + \cosh \beta kv}{\cosh \beta \left( \frac{k^2}{2m} + \frac{q^2}{2m} - \mu \right) + 1} \right] = N \frac{q^2}{2m} + \frac{\Delta^2}{g} - T \sum_k \ln \left[ \frac{\cosh \beta E_k + \cosh \beta kv}{\cosh \beta \bar{\epsilon}_k + 1} \right]. \quad (21)$$

What we should do to obtain the phase diagram is to compare the possible nonzero solution(s) for the order parameter to determine which solution possesses the lower free energy  $F_s(q, \mu)$  and whether this free energy is lower than that of a stationary (i.e., nonmoving) normal state, i.e., whether  $F_s(q, \mu) - F_n(0, \mu) \leq 0$ . For convenience, let us make the definition

$$\delta F(v, \mu) \equiv N \frac{q^2}{2m} - F_s(q, \mu) + F_n(0, \mu) = -\frac{\Delta^2}{g} + T \sum_k \ln \left( \frac{\cosh \beta E_k + \cosh \beta kv}{\cosh \beta \bar{\epsilon}_k + 1} \right). \quad (22)$$

The solution for  $\Delta$  that has the largest value of

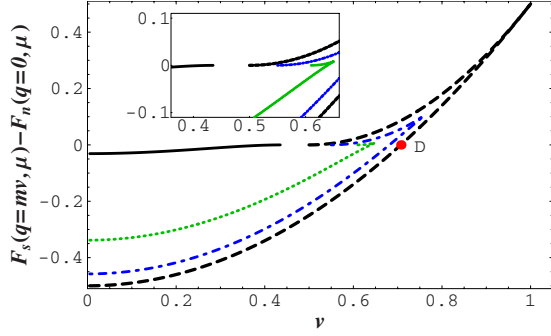


FIG. 3. (Color online) Free-energy difference  $F_s(q=mv, \mu) - F_n(0, \mu) = Nq^2/2m - \delta F$  (in units of  $N_0\Delta_0^2$ ) at various temperatures  $T=(0, 0.2, 0.4, 0.85)T_c^0$  (black dashed, blue dashed dotted, green dotted, black solid, respectively) vs superfluid velocity  $v$  (in units of  $v_L$ ). When  $F_s(q, \mu) - F_n(0, \mu)$  is negative, the corresponding superconducting solution is stable. The red dot (D) indicates the transition point at  $v_c = v_L/\sqrt{2}$  and  $T_c = 0$ ; see also the point D in Fig. 4. The two branches for lower temperatures ( $T=0, 0.2, 0.4$ ) represent double solutions for the order parameter. The range of  $v$  for which a nonzero solution exists shrinks as the temperature increases, and the feature of double solutions disappears at sufficiently high temperatures; see also Fig. 1. The inset shows a blowup in the range of  $[0.35, 0.7]$  for  $v$ .

$\delta F - Nq^2/2m$ , provided this difference is greater than zero, is the true equilibrium solution; otherwise the normal state will be the true equilibrium state. The kinetic flow term  $Nq^2/2m$  can be re-expressed [by relating the total electron number to the density of states (DOS)] as  $N_0\Delta_0^2(v/v_L)^2$ , where  $N_0$  is the density of states per spin, and we have for convenience set the total ‘‘volume’’ of the system to be unity. Note that from Eq. (22) we have at zero temperature

$$\delta F(v, \mu) = N_0 \frac{\Delta^2}{2} + 2 \sum_k \left( kv - E_k + \frac{\Delta^2}{2E_k} \right) \Theta(kv - E_k). \quad (23)$$

This expression also holds for higher-dimensional systems, provided we replace  $k$  by  $\vec{k}$ ,  $v$  by  $\vec{v}$ , and  $kv$  by  $\vec{k} \cdot \vec{v}$ . The effective condensation energy  $\delta F$  for the solution  $\Delta = \Delta_0$  [see the discussion below Eq. (13)] is  $N_0\Delta_0^2/2$ . Balancing  $\delta F$  against the additional energy  $Nq^2/2m = N_0\Delta_0^2(v/v_L)^2$  due to the flow we obtain  $v_c/v_L = 1/\sqrt{2}$ . This corresponds to the Clogston-Chandrasekhar limit in the exchange-field case.<sup>7</sup>

In Fig. 3, we plot  $\delta F$  versus the momentum  $q$  for the two superconducting solutions at several temperatures, as well as  $Nq^2/2m$ . The crossing of the curve representing  $Nq^2/2m$  by the curve representing the superconducting state signifies a discontinuous transition. By finding such transitions at various values of  $v$ , we arrive at the equilibrium phase boundary  $T_c(v)$ , shown as the curve ABD in Fig. 4.

## VII. SUPERCOOLING AND SUPERHEATING

As discussed in Sec. VI, we have discussed the true equilibrium phase-transition curve ABD; see Fig. 4. But what is the physical meaning of the branch BC? It would be a limit

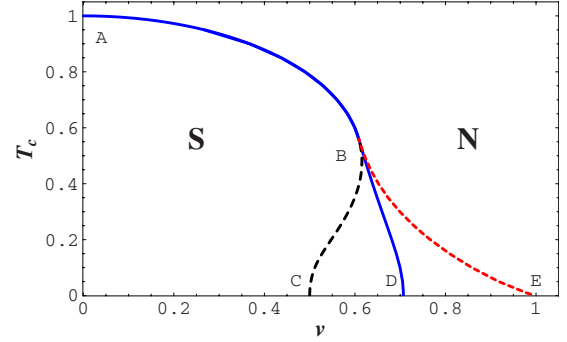


FIG. 4. (Color online) Temperature vs flow-velocity phase diagram, indicating normal (N) and superconducting (S) regions. The phase boundary ABD indicates the transition between superconducting and normal equilibrium states, which is obtained using the principle described below Eq. (21). Across segment AB of the phase boundary the superconducting normal transition is continuous. Across segment BD the transition is discontinuous. As discussed in Sec. VII, line segments BC and BE indicate linear stability limits for, respectively, the normal and superconducting metastable states.  $T_c$  and  $v$  are measured in units of  $T_c^0$  (i.e., the zero-flow critical temperature) and  $v_L$ , respectively.

of metastability if a normal state with flow were a true equilibrium state. Suppose that the system were in an equilibrium normal state with  $v > v_c$  (i.e., lying to the right of the equilibrium phase boundary ABD) and were then ‘‘quenched’’ by the rapid change in  $T$  and/or  $v$  into the metastable region BCD. The system would remain in the normal state, but metastably so, until a fluctuation were to occur that would nucleate a droplet of the superconducting phase large enough to grow and complete the conversion of the state to the true equilibrium state, i.e., the superconducting state. The metastability limit (which is also known as a spinodal line) is formally obtained by solving for the temperature obeying the equation [c.f. Eqs. (10), (15), and (16)]

$$\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \frac{\delta F}{\delta \Delta} = 0. \quad (24)$$

However, as we argue in Sec. VI, a normal state with a constant flow is not a true equilibrium state; therefore, strictly speaking, the curve BC does not represent a meaningful supercooling curve.

On the other hand, the branch BE is a true limit of metastability for the superconducting state. The system can be quenched into a metastable superconducting state from an equilibrium superconducting state [i.e., from  $(T, v)$  to the left of the equilibrium phase boundary] by rapidly changing  $(T, v)$  to a value in the region BED. The system will remain superconducting, but metastably so, until a normal-phase nucleation event carries it to the equilibrium (i.e., normal) state. In this case, the metastability limit is determined by simultaneously solving the equations  $\delta F/\delta \Delta = 0$  and  $\delta^2 F/\delta^2 \Delta = 0$  (but  $\Delta \neq 0$ ) numerically; see Fig. 4.

We note that a corresponding diagram in the case of exchange-field effect in superconductors was first obtained by Maki and Tsuneto.<sup>9</sup> To answer the question of how long it would take for a system in a metastable state to find its

equilibrium state is a kinetic one that would require further investigation.

### VIII. SUPERCURRENT

In this section we examine the dependence of the supercurrent on the flow velocity and thereby determine the critical current, i.e., the maximum equilibrium supercurrent that the system can sustain at various temperatures. In general, the charge current  $I_Q$  carried by the system is defined via

$$I_Q \equiv e \sum_{k,\sigma} \frac{q+k}{m} \langle c_{k+q,\sigma}^\dagger c_{k+q,\sigma} \rangle. \quad (25)$$

As  $\sum_{k,\sigma} \langle c_{k+q,\sigma}^\dagger c_{k+q,\sigma} \rangle$  is the total number of electrons  $N$ , we can rewrite Eq. (25) as

$$I_Q = \frac{e}{m} N v + e \sum_{k,\sigma} \frac{k}{m} \langle c_{k+q,\sigma}^\dagger c_{k+q,\sigma} \rangle, \quad (26)$$

where  $v \equiv q/m$ . By using Eqs. (4) and (8), the second term can be simplified to  $(2e/m) \sum_k k f_{1;k}$ , and thus the expression for  $I_Q$  becomes

$$I_Q(T, v) = -\frac{|e|}{m} \left( N m v + 2 \sum_k k f_{1;k} \right), \quad (27)$$

where the electron charge is  $e = -|e|$  and the temperature  $T$  is implicit in the Fermi function  $f_{1;k}$ ; see Eq. (8).

We shall compute the current in the superconducting state as a function of  $T$  and  $v$ , and then, by choosing the value  $v_m(T)$  of  $v$  in the range  $[0, v_c(T)]$  that maximizes  $I_Q(T, v)$  at fixed  $T$ , we shall obtain the critical current denoted by  $I_c(T) \equiv I_Q[T, v_m(T)]$ , at that temperature. Equation (27), applied at  $T=0$ , tells us that the zero-temperature current is given by

$$I_Q(T=0, v) = -\frac{|e|}{m} [N m v - 2 N_0 k_F \sqrt{k_F^2 v^2 - \Delta_0^2} \Theta(k_F v - \Delta_0)], \quad (28)$$

$$= I_Q^0 \left[ \frac{v}{v_L} - \sqrt{\frac{v^2}{v_L^2} - 1} \Theta(v - v_L) \right], \quad (29)$$

where  $I_Q^0 \equiv -N|e|\Delta_0/k_F = -2|e|\Delta_0/\pi$ . Although, formally,  $I_Q(T=0, v)$  achieves its maximum value (viz.,  $I_Q^0$ ) at  $k_F v = \Delta_0$ , this maximum value is unattainable because the superconducting state gives way to the normal state via a pre-emptive transition at  $k_F v = \Delta_0/\sqrt{2}$  (unless the system falls out of equilibrium and remains metastably in the superconducting state). Therefore, the maximum equilibrium critical current at zero temperature occurs at  $v = v_L/\sqrt{2}$ , and has the value  $I_Q^0/\sqrt{2}$ . At  $v = v_L/\sqrt{2}$  the superconducting state first becomes unstable, so the critical current that is attainable in practice is slightly less than  $I_Q^0/\sqrt{2}$ . This is unlike the behavior in three dimensions as well as the higher of the nonzero temperatures; for these situations the maximum current is achieved at a value of superfluid velocity for which the superconducting state is still globally stable.

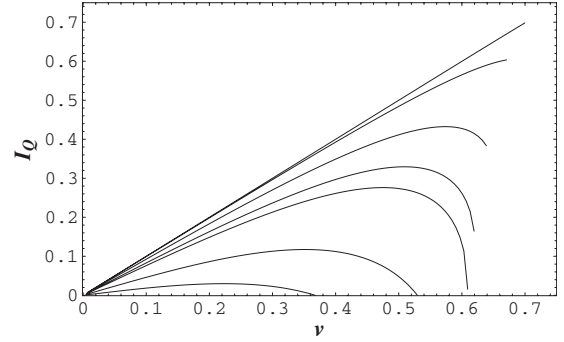


FIG. 5. Supercurrent  $I_Q$  (in units of  $I_Q^0$ ) vs superfluid velocity  $v$  (in unit of  $v_L$ ) for various temperatures. From top to bottom:  $T = (0.1, 0.25, 0.4, 0.5, 0.556, 0.75, 0.9)T_c^0$ . The curves terminate at the critical velocities  $v_c(T)$  appropriate to these temperatures. The maximum supercurrent for a particular curve determines the value of the critical current at that temperature.

In Fig. 5 we show the equilibrium supercurrent  $I_Q$  vs the superfluid velocity  $v$  for various values of the temperature. The value of  $v = v_m$  that corresponds to a maximum in the current (i.e., the critical current  $I_c$ ) is not necessarily  $v_c$ . (It is  $v_c$  for lower temperatures, but not for higher ones.) The temperature dependence of the critical current  $I_c(T)$  is shown in Fig. 6.

A related quantity of interest is the superfluid density  $n_s$ , defined as

$$n_s = \left. \frac{\partial j_s(v)}{\partial v} \right|_{v=0}, \quad (30)$$

i.e., the response of the current density  $j_s$  (whose expression is identical to  $I_Q$  because we have set the system volume to unity) to an infinitesimal change in the ‘driving’ velocity  $v$ . The temperature dependence of  $n_s$  is shown in Fig. 7. The vanishing of  $n_s$  near  $T_c$  is, as expected, linear and the departure of  $n_s$  from its zero-temperature value has, at low temperatures, a thermally activated form.

### IX. EFFECT OF DISORDER

So far, we have discussed clean systems, i.e., systems not disordered by any scattering electrons by impurities. What

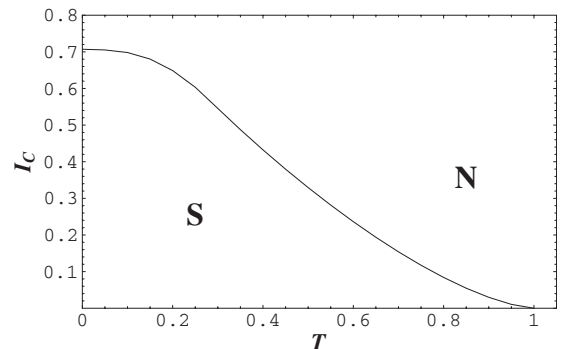


FIG. 6. Critical current  $I_c$  (in units of  $I_Q^0$ ) vs temperature  $T$  (in units of  $T_c^0$ ). This is also current vs temperature phase diagram.

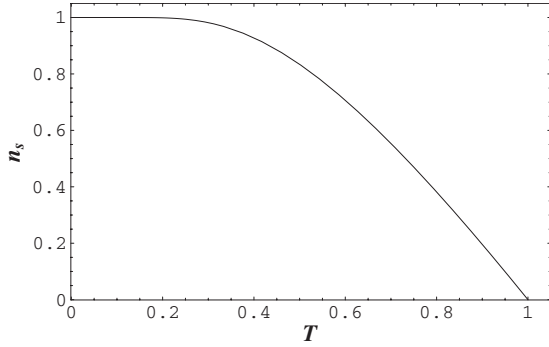


FIG. 7. Superfluid density  $n_s(T)$  (in units of  $2ek_F/\pi$ ) vs temperature  $T$  (in units of  $T_c^0$ ).

effect will disorder have? In particular, would the disorder change the nature of the superconducting-to-normal transition? In this section we briefly discuss the latter issue. In the case of disorder, the Green's function technique is better suited for dealing with disorder than is trying to diagonalize the Hamiltonian with inclusion of an arbitrary configuration of the impurities. An equivalent calculation in the context of the exchange field was done by Maki and Tsuneto<sup>9</sup> and more recently by Wei and Goldbart<sup>18</sup> in the context of Little-Parks effect in small rings. Here, in contrast with Ref. 18, we are not concerned with the finiteness of the system size, and we simply quote the relevant results for the equation obeyed by the critical temperature, i.e., Eq. (B39) of Ref. 18

$$\ln \left[ \frac{T_c(v)}{T_c^0} \right] = \psi \left( \frac{1}{2} \right) - \frac{1}{\sqrt{\alpha^2 - \chi^2}} \times \left[ \frac{-\alpha + \sqrt{\alpha^2 - \chi^2}}{2} \psi \left( \frac{1 + \alpha + \sqrt{\alpha^2 - \chi^2}}{2} \right) + \frac{\alpha + \sqrt{\alpha^2 - \chi^2}}{2} \psi \left( \frac{1 + \alpha - \sqrt{\alpha^2 - \chi^2}}{2} \right) \right], \quad (31)$$

where, in the present case,  $\alpha \equiv 1/4\pi\tau_0 T_c(v)$  and  $\chi \equiv k_F v / \pi T_c(v)$ , the parameter  $\tau_0$  is the elastic mean-free time, and  $\psi(x)$  is the di-gamma function.<sup>16</sup>

In the clean limit (i.e.,  $\tau_0\Delta_0 \gg 1$ ), Eq. (31) reduces to Eq. (17), for which multiple solutions for  $T_c(v)$  exist. For strong disorder ( $\tau_0\Delta_0 \ll 1$ ), Eq. (31) reduces to

$$\ln \left[ \frac{T_c(v)}{T_c^0} \right] = \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1}{2} + \frac{\chi^2}{4\alpha} \right), \quad (32)$$

for which no multiple solutions for  $T_c(v)$  exist. This suggests that the assumption of a vanishing order parameter in the search for the critical temperature is valid and, hence, that the transition is continuous. Moreover, from Eq. (32), in the strong-disorder limit the critical velocity is determined via

$$k_F v_c / \Delta_0 = 1 / \sqrt{4\tau_0\Delta_0}. \quad (33)$$

Numerically, we have found that for  $\tau_0\Delta_0 \leq 0.55$  multiple solutions do not occur. This, then gives the threshold for the disorder strength that divides the discontinuous and continu-

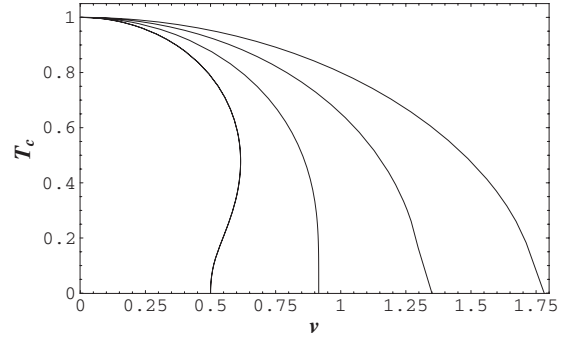


FIG. 8. The impact of elastic scattering on the flow-velocity-dependent normal-to-superconducting transition temperature solutions obtained via the self-consistency, Eq. (31).  $T_c$  and  $v$  are in units of  $T_c^0$  and  $v_L$ , respectively. From left to right:  $\hbar/\tau_0\Delta_0 = (0, 1, 1/0.546, 5, 10)$ , i.e., cleaner to dirtier.

ous transition regimes. In Fig. 8 we show the solutions of Eq. (31) for various values of  $\tau_0\Delta_0$ . In the strong-disorder limit we obtain from Eq. (33) that  $v_c/v_L = \sqrt{\pi\xi_0/4l_e}$  (where  $l_e$  is the elastic mean-free path and  $\xi_0$  is the zero-temperature coherence length), in particular, that the critical velocity exceeds  $v_L$ . But will the critical *current* also exceed its clean-limit value? According to Bardeen,<sup>4</sup> the superfluid density in the strong-disorder limit is reduced by a factor of roughly  $l_e/\xi_0$ . One order-of-magnitude estimate gives for the critical current the product of the superfluid density  $n_s \sim (l_e/\xi_0)n_s^0$  (where  $n_s^0$  denotes the superfluid density in the clean limit) and the critical velocity  $v_c = \sqrt{\pi\xi_0/4l_e}v_L$ . This in turn gives that the critical supercurrent is reduced by roughly a factor  $\sqrt{l_e/\xi_0}$ , which is much less than unity in the strong-disorder limit. Therefore, even though the critical velocity is increased by disorder, the critical current is reduced by it.

We now digress to discuss the scenario of a closed superconducting loop threaded by magnetic flux, which is pertinent to the experiment performed by Liu *et al.*<sup>19</sup> and the situation considered theoretically in Ref. 18. There, the boosted velocity is induced by the magnetic flux. The measurement of  $T_c$  vs flux  $\phi \equiv \Phi/\Phi_0$  (with  $\Phi_0 \equiv hc/2e$ ) presented in Fig. 4 of Ref. 19 shows non- $\Phi_0$ -periodic behavior, with the first dome (near  $\phi=0$ ) being higher than the second dome (near  $\phi=1$ ). Naturally, the data explore only a limited range of flux, so it is not possible to determine with certainty whether higher flux values would yield dome heights that oscillate with flux or decay. Oscillations would be a consequence of flux through the hole; decay would be a consequence of flux through the sample, which would cause pair breaking. To ascertain which of these two effects (oscillation or decay) is likely to dominate in the range of fluxes probed by the experiment, it is useful to compare several length scales. First, the sample radius  $R \approx 75$  nm is smaller than the coherence length  $\xi_0$ , by a factor of roughly 0.06. Following Ref. 18, this suggests the smallness of the radius as the origin of the dome-height mismatch. On the other hand, a more complete characterization of the system would take into account the elastic mean-free path  $l_e$ , which for the sample studied in Ref. 19 is shorter than the radius by a factor of approximately 7.8. This puts the sample in the dirty regime and as a result the impact of the smallness of the radius is



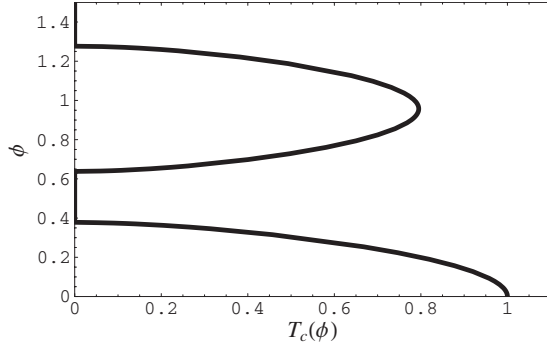


FIG. 9. Critical temperature  $T_c(\phi)$  (in unit of zero-flux critical temperature  $T_c^0$ ) vs magnetic flux [ $\phi \equiv \Phi/(hc/2e)$ ] for a superconducting ring. The result is obtained by using Eq. (34) with the additional pair-breaking term given in Eq. (35), and a numerical factor of about 2.5 is the only fitting parameter. All other parameters are taken from those reported in the experiment by Liu *et al.* (Ref. 19).

suppressed and tends to restore the  $\Phi_0$  periodicity of  $T_c$  discussed in Ref. 18 and given by

$$\ln \left[ \frac{T_c(\phi)}{T_c^0} \right] = \ln t(\phi) = \psi \left( \frac{1}{2} \right) - \psi \left[ \frac{1}{2} + \frac{\Gamma l_e \xi_0 x_m^2(\phi)}{t(\phi) R^2} \right], \quad (34)$$

where  $\Gamma \approx 1.76$  and  $x_m(\phi) = \min_{m \in \mathbb{Z}} |\phi - m|/2$ . What else could cause the lack of  $\Phi_0$  periodicity? Because the magnetic flux is not confined inside the hole in the loop but is also present throughout the sample owing to its finite thickness the orbital pair-breaking effect comes into play. To account for this effect, we use the scenario of the orbital pair-breaking effect of a parallel magnetic field applied to a film of finite thickness  $d$  (which is about 25 nm in the experiment), which leads us to add to the argument of the second term on the rhs of Eq. (32) the term

$$\frac{1}{6} \frac{De^2 B^2 d^2}{2\pi k_B T_c(\phi) \hbar c^2} = \frac{\Gamma \phi^2 \xi_0 l_e d^2}{36 t(\phi) R^4}, \quad (35)$$

see, e.g., Ref. 20. (The term should be correct up to a numerical factor due to geometry.) As we see from Fig. 9, the resulting dependence of the critical temperature on the flux  $\phi$  seems to reproduce, at least qualitatively, the features reported by Liu *et al.*<sup>19</sup>

## X. HIGHER DIMENSIONS

We have seen that in the clean limit the discontinuous transition at flow velocity  $v = \Delta_0/k_F \sqrt{2}$  occurs due to a competition between the condensation energy  $N_0 \Delta_0^2/2$  and the flow kinetic energy  $Nq^2/2m = N_0 \Delta_0^2 (v/v_L)^2$  (for one dimension; see Ref. 21), where  $v_L \equiv \Delta_0/k_F$ . The principle that we employed in Sec. VI is to compare the free energies of the flowing superconducting states and of the stationary normal state. As further confirmation of the validity of this principle, we now apply it to higher-dimensional settings, and compare the results with the known one mentioned in Sec. I.

In two dimensions, the solution of the self-consistency [Eq. (11)] gives  $\Delta = \Delta_0$  for  $v \leq v_L$ , and  $\Delta = 0$  for  $v > v_L$ . This indicates a discontinuous transition at  $v_c^{(2D)} = v_L$ . Indeed, this result is consistent with free-energy considerations: by using the condensation energy  $\delta F = N_0 \Delta_0^2/2$  and the flow kinetic energy  $Nq^2/2m = N_0 \Delta_0^2 (v/v_L)^2/2$  (the difference from the one-dimensional case arising from the density of states; see Ref. 21), we conclude that  $v_c^{(2D)} = v_L$ .

In three dimensions the solution of the self-consistency [Eq. (11)] gives  $\Delta = \Delta_0$  for  $v < v_L$ , and for  $v_L \leq v \leq (e/2)v_L$  (see Ref. 5) gives the following implicit equation for  $\Delta$

$$\sqrt{1 - \left( \frac{\Delta}{k_F v} \right)^2} - \ln \left[ 1 + \sqrt{1 - \left( \frac{\Delta}{k_F v} \right)^2} \right] = \ln \left( \frac{v}{v_L} \right), \quad (36)$$

where we have ignored terms of order  $\Delta/\omega_D$  or smaller (with  $\omega_D$  denoting the Debye frequency). Equation (36) gives that  $\Delta = 0$  at  $v = v_c^{(3D)} = (e/2)v_L \approx 1.359v_L$ . Is this linear-instability result consistent with free-energy considerations (i.e., the balancing of the two energies)? If one were to naively use the condensation energy  $N_0 \Delta_0^2/2$  to balance the flow energy  $Nq^2/2m = N_0 \Delta_0^2 (v/v_L)^2/3$ ,<sup>21</sup> one would obtain  $v_c = \sqrt{3}/2 (\approx 1.225)v_L < v_c^{(3D)}$ ! Does this mean that the superconducting state really becomes unstable at  $v \approx 1.225v_L < v_c^{(3D)}$ ? Such a result would be in conflict with the result mentioned in Sec. I, that  $v_c^{(3D)} = (e/2)v_L$ ? How is this apparent conflict resolved? For  $v \leq v_L$ , the effective condensation energy  $\delta F$  is indeed  $N_0 \Delta_0^2/2$ . But for  $v > v_L$ ,  $\delta F \neq N_0 \Delta_0^2/2$ , so it would be incorrect to equate  $N_0 \Delta_0^2/2$  and  $Nq^2/2m$  in order to deduce  $v_c$ . Instead, one should use the expression for  $\delta F$  appropriate to the range of  $v > v_L$ . In fact, an evaluation of Eq. (23) in three dimensions gives (for  $v > v_L$ )

$$\begin{aligned} \delta F = & \frac{N_0}{2} \Delta^2 + \frac{N_0}{3} (k_F v)^2 \left[ \sqrt{1 - \left( \frac{\Delta}{k_F v} \right)^2} \right]^3 \\ & + N_0 \Delta^2 \sinh^{-1} \frac{\sqrt{(k_F v)^2 - \Delta^2}}{\Delta} \\ & - N_0 \Delta^2 \ln \frac{\sqrt{(k_F v)^2 - \Delta^2} + k_F v}{\Delta}, \end{aligned} \quad (37)$$

where the dependence of  $\Delta$  on  $v$  is given by Eq. (36). From Eqs. (23) and (37) we have that for  $v_L < v < (e/2)v_L$ ,  $\Delta > 0$  and  $\delta F - Nq^2/(2m) > 0$ . Only when  $v$  increases to  $v = (e/2)v_L$  (i.e., when  $\Delta = 0$ ) do we have  $\delta F - (Nq^2/2m) = 0$ . Thus, the transition at  $v = (e/2)v_L$  is continuous, as the order parameter vanishes continuously there and, hence, the conflict is resolved. The dependence of the quantity  $F_s(q, \mu) - F_n(0, \mu) = Nq^2/(2m) - \delta F$  on flow velocity  $v$  is plotted in Fig. 10, which clearly shows that the superconducting state is stable for flow velocities in the range of  $0 \leq v < (e/2)v_L$ .

## XI. CONCLUDING REMARKS

We have revisited the issue of the critical velocity of an effectively one-dimensional superconductor using mean-field

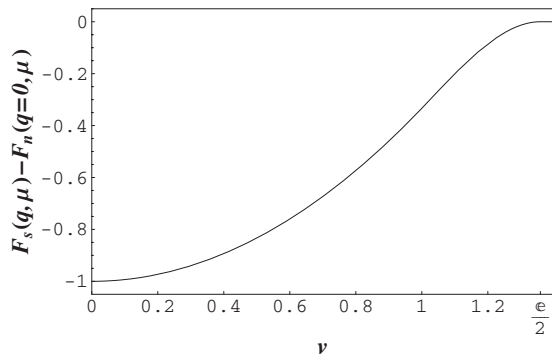


FIG. 10. The energy difference  $F_s(q=mv, \mu) - F_n(0, \mu)$  (in units of  $N_0 \Delta_0^2/2$ ) vs flow velocity  $v$  (in units of  $v_L$ ) in three spatial dimensions. When the difference is less than zero, the flowing superconducting state is stable.

theory. As we have discussed, this issue is equivalent to that of the threshold uniform exchange field in a superconductor, first investigated by Sarma<sup>8</sup> and by Maki and Tsuneto.<sup>9</sup> This equivalence is due to the correspondence of (i) the Zeeman frustration energy  $\mu_B B$  between the two electrons of a Cooper pair (in the case of exchange field) and (ii) the kinetic frustration energy  $2k_F v$  (in the case of flow). In particular, we have found that at zero temperature the critical velocity is a factor of  $\sqrt{2}$  smaller than the Landau critical velocity, in contrast to a previous finding.<sup>6</sup> Our result originates in a Clogston-Chandrasekhar-type discontinuous transition between the superconducting and the normal state. The physical reason for the discontinuity of the transition is that it results from a balance between the condensation and flow energies. (Recall that the bulk superconducting phase transition in magnetic field results from a balance between the condensation ( $N_0 \Delta_0^2/2$ ) and magnetic-field ( $B^2/8\pi$ ) energies, and that the transition is discontinuous.) The transition re-

mains discontinuous for temperatures below approximately  $0.56T_c^0$ ; above that it is continuous.

We have also studied the issue of critical supercurrents, determined the current-temperature phase diagram, and examined metastability (and its limits) in the temperature versus flow-velocity phase diagram. As a test of our underlying principle, namely, the comparison of the free energies of flowing superconducting and stationary normal states, we have also examined the two- and three-dimensional cases, and have thus obtained results that are in agreement with previous findings. In addition, we have commented on the effects of electron scattering by impurities and, in particular, we have argued that strong disorder renders continuous the aforementioned discontinuous phase transition. Even though disorder can increase the value of the critical *velocity*, the physically measurable quantity, i.e., the critical current, is still reduced by the disorder.

Throughout this paper, we have adopted a mean-field approach, thus ignoring the effects of fluctuations such as phase slips. Furthermore, we have limited ourselves to the analysis of equilibrium properties as well as issues of metastability. The intriguing issue of how a flowing superconducting state undergoes the transition to the stationary normal state, which essential dynamical processes take place, and on what time-scale all remain as research directions for the future.

#### ACKNOWLEDGMENTS

The authors acknowledge useful discussions with Anton Burkov, Tony Leggett, and Frank Wilhelm. This work was supported by IQC, NSERC, and ORF (TCW) and by DOE under Grant No. DEFG02-07ER46453 through the Frederick Seitz Materials Research Laboratory at the University of Illinois (PMG).

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